1. Show that $\lim_{x \to 0} \cos \frac{1}{x}$ does not exist.

2. Suppose $f, g: \mathbb{R} \to \mathbb{R}$ and $x_0, y_0, l \in \mathbb{R}$. If

   (i) $\lim_{x \to x_0} g(x) = y_0$ and $\lim_{y \to y_0} f(y) = l$; and

   (ii) there exists $\delta > 0$ s.t. $g(x) \neq y_0$ if $x \in \mathbb{R}$, $0 < |x - x_0| < \delta$

   Show that $\lim_{x \to x_0} f(g(x)) = l$. Can we drop condition (ii)?

3. Prove the squeeze Theorem: Let $A \subseteq \mathbb{R}$, let $f, g, h: A \to \mathbb{R}$. Let $c \in \mathbb{R}$ be a cluster point of A. If

   $\forall x \in A, x \neq c$, $f(x) \leq g(x) \leq h(x)$ and $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$

   Show that $\lim_{x \to c} g(x) = L$

4. Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, and c be a cluster point of both of the sets $A \cap (c, \infty)$ and $A \cap (-\infty, c)$.

   Show that $\lim_{x \to c} f(x) = L$ iff $\lim_{x \to c^+} f(x) = L$ and $\lim_{x \to c^-} f(x) = L$.

5 a) State the definition of limits at infinity.

   b) Evaluate $\lim_{x \to \infty} \frac{\sqrt{x} - x}{\sqrt{x} + x}$ (if exist) by definition.

6. Let $f: (0, \infty) \to \mathbb{R}$. Prove that

$$\lim_{x \to \infty} f(x) = L \quad \text{iff} \quad \lim_{x \to 0^+} f\left(\frac{1}{x}\right) = L$$

1. By Thm 4.1.9, only need to find two sequence (x_n), (y_n)

with $x_n \neq 0$, $y_n \neq 0$ $\forall n$ s.t.

$$\lim_n x_n = 0 = \lim_n y_n \quad \text{but} \quad \lim_n \cos\frac{1}{x_n} \neq \lim_n \cos\frac{1}{y_n}$$

Let $x_n = \frac{1}{2n\pi}$, $y_n = \frac{1}{2n\pi + \pi}$

Then $\lim x_n = 0 = \lim y_n$

$$\lim \cos\frac{1}{x_n} = \lim \cos 2n\pi = 1$$

$$\lim \cos\frac{1}{y_n} = \lim \cos(2n\pi + \pi) = -1 \neq \lim \cos\frac{1}{x_n}$$

Hence, $\lim_{x \to 0} \cos\frac{1}{x}$ does not exist.

2. $\lim_{x \to x_0} f(g(x)) = l \iff \forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $x \in \mathbb{R} \setminus \{x_0\}$, $|x - x_0| < \delta$,

then $|f(g(x)) - l| < \varepsilon$

Let $\varepsilon > 0$.

condition (a)

$\left(\lim_{y \to y_0} f(y) = l \Rightarrow\right)$ $\exists \gamma > 0$ s.t. if $\boxed{y \in \mathbb{R} \setminus \{y_0\},\ |y - y_0| < \gamma}$,

then $|f(y) - l| < \varepsilon$

$\left(\lim_{x \to x_0} g(x) = y_0 \Rightarrow\right)$ for $\gamma > 0$, $\exists \delta'$ s.t. if $x \in \mathbb{R} \setminus \{x_0\}$, $|x - x_0| < \delta'$,

then $|g(x) - y_0| < \gamma$

$g(x)$ not satisfy condition (a), $g(x)$ may be $= y_0$

use (ii) $\exists \delta'' > 0$ s.t. $g(x) \neq y_0$ if $x \in \mathbb{R}$, $0 < |x - x_0| < \delta''$

$\Downarrow$

$x \in \mathbb{R} \setminus \{x_0\}$, $|x - x_0| < \delta''$

Choose $\delta = \min\{\delta', \delta''\}$, if $x \in \mathbb{R} \setminus \{x_0\}$, $|x - x_0| < \delta$

$|g(x) - y_0| < \gamma$ and $g(x) \neq y_0 \Rightarrow g(x)$ satisfy condition (a)

$\Rightarrow |f(g(x)) - l| < \varepsilon$.

2. Condition (ii) is important.

If we drop condition (ii),

if $g(x) = y_0 \quad \forall x \in \mathbb{R}$

$g(x)$ cannot satisfy condition (a)

Then $\lim_{x \to x_0} f(g(x)) \neq \ell$.

3. if $x \neq c$, $x \in A$

$$f(x) \leq g(x) \leq h(x)$$

$$f(x) - L \leq g(x) - L \leq h(x) - L$$

Let $\varepsilon > 0$, $\exists\, \delta_1 > 0$ s.t.: if $x \in A \setminus \{c\}$, $|x - c| < \delta_1$,

then $|f(x) - L| < \varepsilon \Rightarrow f(x) - L > -\varepsilon$

$\exists\, \delta_2 > 0$ s.t. if $x \in A \setminus \{c\}$, $|x - c| < \delta_2$,

then $|h(x) - L| < \varepsilon \Rightarrow h(x) - L < \varepsilon$

Choose $\delta = \min\{\delta_1, \delta_2\}$, if $x \in A \setminus \{c\}$, $|x - c| < \delta$

$\Rightarrow |x - c| < \delta_1$, $|x - c| < \delta_2$

Then $-\varepsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \varepsilon$

$$\Rightarrow |g(x) - L| < \varepsilon$$

Hence $\lim_{x \to c} g(x) = L$.

4. " $\Rightarrow$ " Suppose $\lim_{x \to c} f(x) = L$

Let $\varepsilon > 0$, $\exists \delta > 0$ s.t. if $x \in A$, $0 < |x-c| < \delta$,

then $|f(x) - L| < \varepsilon$

Choose this δ, if $x \in A$, $0 < x-c < \delta$

$\Rightarrow 0 < |x-c| < \delta$

$\Rightarrow |f(x) - L| < \varepsilon$

$\Rightarrow \lim_{x \to c^+} f(x) = L$

if $x \in A$, $0 < c-x < \delta$

$\Rightarrow 0 < |x-c| < \delta$

$\Rightarrow |f(x) - L| < \varepsilon$

$\Rightarrow \lim_{x \to c^-} f(x) = L$

" $\Leftarrow$ " Suppose $\lim_{x \to c^+} f(x) = L = \lim_{x \to c^-} f(x)$

Let $\varepsilon > 0$, by definition $\left(\lim_{x \to c^+} f(x) = L\right) \Rightarrow$ $\exists \delta_1 > 0$ s.t. if $x \in A$, $0 < x-c < \delta_1$,

then $|f(x) - L| < \varepsilon$

$\left(\lim_{x \to c^-} f(x) = L\right) \Rightarrow \exists \delta_2 > 0$ s.t. if $x \in A$, $0 < c-x < \delta_2$,

then $|f(x) - L| < \varepsilon$

Choose $\delta = \min\{\delta_1, \delta_2\}$, if $x \in A$, $0 < |x-c| < \delta$,

Case 1: $x > c$, $0 < x-c < \delta < \delta_1$, $|f(x) - L| < \varepsilon$

Case 2: $x < c$, $0 < c-x < \delta < \delta_2$, $|f(x) - L| < \varepsilon$

Hence, $\lim_{x \to c} f(x) = L$.

5(a) Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$.

Suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$

L is said to be limit of f as $x \to \infty$

write $\lim_{x\to\infty} f(x) = L$,

if $\forall \varepsilon > 0$, $\exists K > a$ such that $\forall x > K$, $|f(x) - L| < \varepsilon$

b)

Observe that $\lim_{x\to\infty} \frac{\sqrt{x} - x}{\sqrt{x} + x} = \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}} - 1}{\frac{1}{\sqrt{x}} + 1} = -1$

Want to show $\left| \frac{\sqrt{x} - x}{\sqrt{x} + x} + 1 \right| < \varepsilon$

$$\left| \frac{\sqrt{x} - x}{\sqrt{x} + x} + 1 \right| = \left| \frac{2\sqrt{x}}{\sqrt{x} + x} \right|$$

$$= \frac{2}{1 + \sqrt{x}} \quad \text{if } x > 0$$

$$\leq \frac{2}{\sqrt{x}}$$

Let $\varepsilon > 0$

since $\frac{1}{x}$ define on $(0, \infty)$

Since $\lim_{x\to\infty} \frac{1}{x} = 0$, $\exists K > 0$ s.t. $\forall x > K$, $\frac{1}{x} < \frac{\varepsilon^2}{4} \Rightarrow \frac{1}{\sqrt{x}} < \frac{\varepsilon}{2}$.

(easy to prove)

Choose above $K > 0$, if $x > K$,

$$\left| \frac{\sqrt{x} - x}{\sqrt{x} + x} + 1 \right| \leq \frac{2}{\sqrt{x}} < \varepsilon$$

Then $\lim_{x\to\infty} \frac{\sqrt{x} - x}{\sqrt{x} + x} = -1$

6. "$\Rightarrow$": Suppose $\lim_{x\to\infty} f(x) = L$

let $\varepsilon > 0$, $\exists K > 0$ s.t. if $\boxed{x > K}$, condition (a)

then $|f(x) - L| < \varepsilon$

Choose $\delta = \frac{1}{K} > 0$, if $y \in (0,\infty)$, $0 < y - 0 < \delta$ (Want $|f(\frac{1}{y}) - L| < \varepsilon$)

$0 < y < \delta \Rightarrow 0 < y < \frac{1}{K}$

$\Rightarrow \frac{1}{y} > K > 0$

$\frac{1}{y}$ satisfy condition (a)

$\Rightarrow |f(\frac{1}{y}) - L| < \varepsilon$

$\Rightarrow \lim_{y\to 0^+} f(\frac{1}{y}) = L$

"$\Leftarrow$". Suppose $\lim_{y\to 0^+} f(\frac{1}{y}) = L$

let $\varepsilon > 0$, $\exists \delta > 0$, if $y \in (0,\infty)$, $0 < y - 0 < \delta$

then $|f(\frac{1}{y}) - L| < \varepsilon$

Choose $K = \frac{1}{\delta} > 0$, if $x > K$ (>0)

$\Rightarrow \frac{1}{x} < \frac{1}{K} = \delta$

$\Rightarrow \frac{1}{x} \in (0,\infty)$ and $0 < \frac{1}{x} - 0 < \delta$

$\Rightarrow |f(\frac{1}{\frac{1}{x}}) - L| < \varepsilon$

$\Rightarrow |f(x) - L| < \varepsilon$

Thus $\lim_{x\to\infty} f(x) = L$.